Supplementary material for: A Bayesian additive model for understanding public transport usage in special events

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1 Moments of a one-side truncated Gaussian

In order to make this result more general, we will derive these moments using a general $\mathcal{N}(x|\mu, \sigma^2)$ and a lower threshold l. The normalization constant, Z, of this one-side truncate Gaussian is given by

$$Z = \int \mathbb{I}(x > l)\mathcal{N}(x|\mu, \sigma^2) dx \tag{1}$$

$$= \int_l^{+\infty} \mathcal{N}(x|\mu, \sigma^2) dx = \Phi\Big(\frac{\mu - l}{\sigma}\Big) \tag{2}$$

where $\Phi(a)$ denotes the value of the cumulative distribution functions (CDF) of a Gaussian distribution evaluated at a. Differentiating both sides w.r.t. μ gives

$$\int_l^{+\infty} \frac{\partial \mathcal{N}(x|\mu, \sigma^2)}{\partial \mu} dx = \frac{\partial \Big(\Phi(\frac{\mu - l}{\sigma})\Big)}{\partial \mu} \tag{3}$$

$$\Leftrightarrow \int_l^{+\infty} \Big(\frac{x - \mu}{\sigma^2}\Big)\mathcal{N}(x|\mu, \sigma^2) dx = \frac{1}{\sigma}\mathcal{N}\Big(\frac{\mu - l}{\sigma}\Big) \tag{4}$$

where we made use of the fact that $\frac{\partial \Phi(z)}{\partial \mu} = \mathcal{N}(z)\frac{\partial z}{\partial \mu}$. Continuing developing the expression gives

$$\Leftrightarrow \frac{1}{\sigma^2}\int_l^{+\infty} x\mathcal{N}(x|\mu,\sigma^2)dx - \frac{\mu}{\sigma^2}\int_l^{+\infty} \mathcal{N}(x|\mu,\sigma^2)dx = \frac{1}{\sigma}\mathcal{N}\Big(\frac{\mu-l}{\sigma}\Big) \tag{5}$$

$$\Leftrightarrow \int_l^{+\infty} x\mathcal{N}(x|\mu,\sigma^2)dx - \mu\int_l^{+\infty} \mathcal{N}(x|\mu,\sigma^2)dx = \sigma\mathcal{N}\Big(\frac{\mu-l}{\sigma}\Big) \tag{6}$$

$$\Leftrightarrow \underbrace{\int_l^{+\infty} x\mathcal{N}(x|\mu,\sigma^2)dx}_{=Z\cdot\mathbb{E}[x]} - \mu\underbrace{\Phi\Big(\frac{\mu-l}{\sigma}\Big)}_{=Z} = \sigma\mathcal{N}\Big(\frac{\mu-l}{\sigma}\Big) \tag{7}$$

$$\Leftrightarrow \mathbb{E}[x]\Phi\Big(\frac{\mu-l}{\sigma}\Big) - \mu\Phi\Big(\frac{\mu-l}{\sigma}\Big) = \sigma\mathcal{N}\Big(\frac{\mu-l}{\sigma}\Big) \tag{8}$$

$$\Leftrightarrow \mathbb{E}[x] = \mu + \sigma\frac{\mathcal{N}(\frac{\mu-l}{\sigma})}{\Phi(\frac{\mu-l}{\sigma})} \tag{9}$$

In order to determine the second moment, we start by differentiating both sides of eq. 1 twice w.r.t. μ:

$$\int_l^{+\infty} \frac{\partial^2\mathcal{N}(x|\mu,\sigma^2)}{\partial\mu^2}dx = \frac{1}{\sigma}\frac{\partial\mathcal{N}(\frac{\mu-l}{\sigma})}{\partial\mu} \tag{10}$$

$$\Leftrightarrow \int_l^{+\infty} \frac{\partial^2\mathcal{N}(x|\mu,\sigma^2)}{\partial\mu^2}dx = -\frac{1}{\sigma}\Big(\frac{\mu-l}{\sigma^2}\Big)\mathcal{N}\Big(\frac{\mu-l}{\sigma}\Big) \tag{11}$$

where we made use of the fact that $\frac{\partial\mathcal{N}(x|\mu,\sigma^2)}{\partial\mu} = -(\frac{x-\mu}{\sigma^2})\mathcal{N}(x|\mu,\sigma^2)$. Continuing differentiating, we get

$$\Leftrightarrow \int_l^{+\infty} \frac{x^2 - 2\mu x + \mu^2 - \sigma^2}{\sigma^4} \mathcal{N}(x|\mu, \sigma^2) dx = -\frac{1}{\sigma} \frac{\mu - l}{\sigma^2} \mathcal{N}\Big(\frac{\mu - l}{\sigma}\Big) \tag{12}$$

$$\Leftrightarrow \int_l^{+\infty} (x^2 - 2\mu x + \mu^2 - \sigma^2) \mathcal{N}(x|\mu, \sigma^2) dx = -\sigma(\mu - l) \mathcal{N}\Big(\frac{\mu - l}{\sigma}\Big) \tag{13}$$

$$\Leftrightarrow \underbrace{\int_l^{+\infty} x^2 \mathcal{N}(x|\mu, \sigma^2) dx}_{=Z\mathbb{E}[x^2]} - 2\mu \underbrace{\int_l^{+\infty} x \mathcal{N}(x|\mu, \sigma^2) dx}_{=Z\mathbb{E}[x]} \tag{14}$$

$$+ (\mu^2 - \sigma^2) \underbrace{\int_l^{+\infty} \mathcal{N}(x|\mu, \sigma^2) dx}_{=Z} = -\sigma(\mu - l) \mathcal{N}\Big(\frac{\mu - l}{\sigma}\Big) \tag{15}$$

$$\Leftrightarrow \mathbb{E}[x^2] Z - 2\mu \mathbb{E}[x] Z + (\mu^2 - \sigma^2) Z = -\sigma(\mu - l) \mathcal{N}\Big(\frac{\mu - l}{\sigma}\Big) \tag{16}$$

$$\Leftrightarrow \mathbb{E}[x^2] - 2\mu \mathbb{E}[x] + \mu^2 - \sigma^2 = -\sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} \tag{17}$$

$$\Leftrightarrow \mathbb{E}[x^2] - 2\mu \Bigg(\mu + \sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg) + \mu^2 - \sigma^2 = -\sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} \tag{18}$$

$$\Leftrightarrow \mathbb{E}[x^2] - \mu^2 + 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \sigma^2 = -\sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} \tag{19}$$

$$\Leftrightarrow \mathbb{E}[x^2] = \mu^2 + \sigma^2 - \sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} \tag{20}$$

$$\Leftrightarrow \mathbb{E}[x^2] = \mu^2 + \sigma^2 \Bigg(1 - \frac{(\mu - l)}{\sigma} \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg) - 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} \tag{21}$$

We can now make use of the two first moments in order to determine the variance, which is given by

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \tag{22}$$

$$= \mu^2 + \sigma^2 \Bigg(1 - \frac{(\mu - l)}{\sigma} \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg) - 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \Bigg(\mu + \sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg)^2 \tag{23}$$

$$= \mu^2 + \sigma^2 - \sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \mu^2 + 2\mu\sigma \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \sigma^2 \Bigg(\frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg)^2 \tag{24}$$

$$= \sigma^2 - \sigma(\mu - l) \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \sigma^2 \Bigg(\frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg)^2 \tag{25}$$

$$= \sigma^2 \Bigg(1 - \frac{(\mu - l)}{\sigma} \frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})} - \Bigg(\frac{\mathcal{N}(\frac{\mu - l}{\sigma})}{\Phi(\frac{\mu - l}{\sigma})}\Bigg)^2\Bigg) \tag{26}$$